Centrally Prime Rings which are Commutative

Adil K. Jabbar
College of Science-University of Sulaimani

Abstract

In this paper the definition of centrally prime rings is introduced, our main purpose is to classify those centrally prime rings which are commutative and so that several conditions are given each of which makes a centrally prime ring commutative.

The Fundamentals

Let \( R \) be a ring. A non-empty subset \( S \) of \( R \) is said to be a multiplicative closed set in \( R \) if \( a, b \in S \) implies \( ab \in S \), and a multiplicative closed set \( S \) is called a multiplicative system if \( [S, R] = \{0\} \), where \( [S, R] = \{ [s, r] : s \in S, r \in R \} \) and \( [s, r] \) is the commutator defined by \( sr - rs \). Define a relation \( (\sim) \) on \( R \times S \) as follows:

If \( (a, s), (b, t) \in R \times S \) then \( (a, s) \sim (b, t) \) iff \( \exists x \in S \) such that \( x(at - bs) = 0 \). Since \( [S, R] = \{0\} \), it can be shown that \( (\sim) \) is an equivalence relation on \( R \times S \). Now denote the equivalence class of \( (a, s) \) in \( R \times S \) by \( a_s \), that is \( a_s = \{(b, t) \in R \times S : (a, s) \sim (b, t)\} \) (this equivalence class is also denoted by \( a \downarrow s \) (Larsen & McCarthy, 1971) or by \( s^{-1}a \)), and then denote the set of all equivalence classes determined under this equivalence relation by \( R_s \), that is let \( R_s = \{a_s : (a, s) \in R \times S\} \). Note that \( R_s \) is also denoted by \( S^{-1}R \) (Larsen & McCarthy, 1971; Ranicki, 2006).

On \( R \times S \) we define addition (+) and multiplication (.) as follows:

\[ a_s + b_t = (at + bs)_{st} \]
\[ a_s \cdot b_t = (ab)_{st}, \forall a_s, b_t \in R_s. \]
It can be shown that these two operations are well-defined and that 
\((R_S, +, \cdot)\) forms a ring which is known as the localization of \(R\) at 
\(S\) , (Fahr, 2002).

Now we mention to some basic definitions:

Let \(R\) be a ring. Then:

\(R\) is called a prime ring if \(aRb = \{0\}\) for \(a, b \in R\) then \(a = 0\) or \(b = 0\)
(Herstein, 1969, Tsai, 2004 and Ashraf, 2005), where \(aRb = \{arb : r \in R\}\).

An additive mapping \(D: R \rightarrow R\) is called a derivation on \(R\) if
\(D(ab) = D(a)b + aD(b), \forall a, b \in R\) (Martindale & Miers, 1983; Vukman, 1999
and Jung & Park, 2006), in other words a mapping \(D: R \rightarrow R\) is called a
derivation on \(R\) if:

1- \(D(a + b) = D(a) + D(b)\), and

2- \(D(ab) = aD(b) + D(a)b, \forall a, b \in R\).

An element \(r \in R\) is called a zero divisor if \(rx = 0\) or \(xr = 0\) for some
nonzero \(x\) in \(R\), and a zero divisor is called a proper zero divisor if it is
nonzero. By the center of \(R\) we mean the set 
\(Z(R) = \{x \in R : xr = rx, \forall r \in R\}\), it can be shown that \(Z(R)\) is a subring
of \(R\).

**Some Remarks**

If \(R\) is a ring and \(S\) is a multiplicative system in \(R\) such that
\([S, R] = \{0\}\), then:

i) \(R_S\) has the identity element though \(R\) does not have , in fact if \(s \in S\)
then \(s_s\) is the identity element of \(R_S\), since if \(a_t\) is any element of \(R_S\),
then \(ast = ats\) which gives \(ast - ats = 0\), then \(s(ast - ats) = 0\) and hence
\(s((as)t - a(ts)) = 0\), thus \((as, ts) \sim (a, t)\) which means \((as)_{ts} = a_t\) or
\(a_t s_s = a_t\) and using the same technique we can show that \(s_s a_t = a_t\)
which means \(\forall s \in S, s_s\) is the identity element of \(R_S\). Note that this
identity does not depend on the choice of elements of \(S\) that is
\(s_s = t_t, \forall s, t \in S\). To prove this, since we have \([S, R] = \{0\}\), so
\([s, t] = \{0\}, \forall s \in S, t \in R\) and hence we get \(st - ts = 0, \forall s \in S, t \in R,\) or
\(st - ts = 0, \forall s, t \in S\), (since \(S \subseteq R\)) then we get \(s(st - ts) = 0, \forall s, t \in S\)
which means \((s, s) \sim (t, t), \forall s, t \in S\), and thus \(s_S = t_S, \forall s, t \in S\).

ii) If \(a, b \in R\) and \(s \in S\), then \(a_S + b_S = (a + b)_S\), to show this we have \((as + bs)s = (a + b)ss\) or \((as + bs)s - (a + b)ss = 0\), which means \(s[(as + bs)s - (a + b)ss] = 0\), and hence \((as + bs, ss) \sim (a + b, s)\), that means \((as + bs)_{ss} = (a + b)_S\) and then we get \(a_S + b_S = (as + bs)_{ss} = (a + b)_S\).

iii) \(\forall s \in S, 0_s\) is the zero of the ring \(R_S\), since if \(a_t \in R_S\) is any element, where \(a \in R, t \in S\), then \(a_t + 0_s = (as + 0t)_{ts} = (as)_{ts} = a_t s_S = a_t\).

Similarly it can be shown that \(0_s + a_t = a_t\) (we will denote \(0_s\) simply by 0).

It is necessary to mention that \(0_S = 0_t, \forall s, t \in S\). That is the zero of \(R_S\) also does not depend on the choice of the elements of \(S\), to prove this it is known that \(0t = 0 = 0s, \forall s, t \in S\), hence we get \(s(0t - 0s) = 0, \forall s, t \in S\) and thus \((0, s) \sim (0, t), \forall s, t \in S\) which means \(0_S = 0_t, \forall s, t \in S\).

iv) If \(a_s \in R_S\), where \(a \in R\) and \(s \in S\) then \((-a)_S\) is the additive identity of \(a_s\) in \(R_S\), that is \(-a_s = (-a)_S\). Note that \(a \in R \Rightarrow -a \in R\) and \(s \in S \Rightarrow (-a)_S \in R_S\) and then \(a_S + (-a)_S = (as + (-a)s)_{ss} = 0_{ss} = 0_s = 0\).

v) If \(a_s = 0\) in \(R_S\), where \(a \in R, s \in S\), then \(\exists t \in S\) such that \(ta = 0\). To show this, we have \(a_s = 0 = 0_s\), hence \((a, s) \sim (0, s)\) which means \(\exists x \in S\) such that \(x(as - 0s) = 0\) or \(xas = 0\). If we let \(t = xs\) then \(x, s \in S\) implies \(t = xs \in S\) and then \(ta = xsa = xas = 0\).

vi) If \(D : R \rightarrow R\) is a mapping then by \(D^2\) we mean \(D \circ D\). In general \(D^n\) will mean \(D \circ D \circ ... \circ D\) (\(n\) times) and finally if \(x \in R\) then by \(xD\) we mean the mapping \(xD : R \rightarrow R\) which is defined by \((xD)(r) = x(D(r)), \forall r \in R\).
The Main Results

Our main purpose in this paper is to transfer some results on prime rings to centrally prime rings, in fact these results known as the commutativity conditions for prime rings and here we will prove the alternative results for centrally prime rings, that is we will determine the conditions which make centrally prime rings commutative but first it is necessary to indicate to the following theorems:

**Theorem A:** (Chung et al., 1979)

Let $R$ be a prime ring and $D : R \to R$ is a derivation on $R$. If $D(x) \in Z(R), \forall x \in R$ then either $D = 0$ or $R$ is commutative.

**Theorem B:** (Felzenszwalb & Giambruno, 1982)

Let $R$ be a prime ring and $U$ is a nonzero ideal of $R$. If $D : R \to R$ is a nonzero derivation on $R$ such that $D(u)u = uD(u), \forall u \in U$ then $R$ is commutative.

**Theorem C:** (Lee & Lee, 1986)

Let $R$ be a prime ring and $I$ a nonzero ideal of $R$. Suppose that $D : R \to R$ is a derivation on $R$ and $n$ is a positive integer such that $D^n(I) \subseteq Z(R)$ then either $D^n = 0$ or $R$ is commutative.

**Theorem D:** (Daif & Bell, 1992)

Let $R$ be a prime ring and $D : R \to R$ is a derivation on $R$. If there exists a nonzero ideal $K$ of $R$ such that either $xy + D(xy) = yx + D(yx), \forall x, y \in K$ or $xy - D(xy) = yx - D(yx), \forall x, y \in K$, then $R$ is commutative.

**Theorem E:** (Bresar, 1993)

Let $R$ be a prime ring and $U$ is a nonzero left ideal of $R$, and suppose that $D, G : R \to R$ are derivations on $R$ satisfying $D(u)u - uG(u) \in Z(R), \forall u \in U$. If $D \neq 0$ then $R$ is commutative.

**Theorem F:** (Filipps, 1999)

Let $R$ be a prime ring and $I$ is a nonzero ideal of $R$. If $D : R \to R$ is a nonzero derivation on $R$ such that $D([x, y]) = [x, y], \forall x, y \in I$ then $R$ is commutative.

Now we introduce the following definitions.
**Definitions**

Let $R$ be a ring. Then we say:
1. $R$ is a centrally prime ring if $R_S$ is a prime ring for each multiplicative system $S$ in $R$ with $[S, R] = \{0\}$.
2. $R$ satisfies central commutation property (CCP) if $R_S$ is commutative for each multiplicative system $S$ in $R$ with $[S, R] = \{0\}$.
3. A derivation $D : R \to R$ is centrally-zero derivation on $R$ if $D(S) = \{0\}$ for each multiplicative system $S$ in $R$ with $[S, R] = \{0\}$.

Before giving the main results of this paper we will prove the following two lemmas which play the basic role in the proof of results of the present paper. In fact the first lemma proves that if $R$ is a ring and $S$ is a multiplicative system in $R$ with $[S, R] = \{0\}$ then each centrally-zero derivation on $R$ induces some derivation on $R_S$.

**Lemma 1:**

Let $R$ be a ring and $S$ a multiplicative system in $R$ such that $[S, R] = \{0\}$.

If $D : R \to R$ is a centrally-zero derivation on $R$, then $D_* : R_S \to R_S$ defined by $D_*(r_S) = (D(r))_S$, $\forall r_S \in R_S$, is a derivation on $R_S$.

**Proof:**

First to show $D_*$ is a mapping. It is clear that $\forall r \in R$ we have $D(r) \in R$ and thus $D_*(r_S) = (D(r))_S \in R_S$, $\forall r_S \in R_S$. Now let $a_t = b_s$ for $a_t, b_s \in R_S$, then $(a, t) - (b, s)$.

Hence there exists $x \in S$ such that $x(as - bt) = 0$. So we have $xas = xbt$ or $xsa = xtb$ (since $[a, s] = 0$ and $[b, t] = 0$) and thus $D(xsa) = D(xtb)$ or $(xs)D(a) + D(xs)a = (xt)D(b) + D(xt)b$, then we get $(xs)D(a) = (xt)D(b)$ (since $D(xs) = 0 = D(xt)$). Hence $x(sD(a) - tD(b)) = 0$ which yields $x(D(a)s - D(b)t) = 0$ (since $[S, R] = \{0\}$) and so $(D(a), t) - (D(b), s)$.

Hence $(D(a))_t = (D(b))_s$ and thus $D_*(a_t) = D_*(b_s)$ which means $D_*$ is a mapping. Now we will show that:
1- \( D_*(a_s + b_t) = D_*(a_s) + D_*(b_t) \) and

2- \( D_*(a_s b_t) = a_s D_*(b_t) + D_*(a_s) b_t \), \( \forall a_s, b_t \in R_S \)

For the proof of the first we have

\[
D_*(a_s + b_t) = D_*((at + bs)_{st}) = (D(at + bs))_{st}
\]

\[
= (D(at) + D(bs))_{st} = (D(at))_{st} + (D(bs))_{st} = (D(a)t)_{st} + (D(b)s)_{ts}
\]

\[
(D(a))_s t_t + (D(b))_t s_s = (D(a))_s + (D(b))_t = D_*(a_s) + D_*(b_t).
\]

Also we have

\[
D_*(a_s b_t) = D_*((ab)_{st}) = (D(ab))_{st} = (aD(b) + D(a)b)_{st} =
\]

\[
(aD(b))_{st} + (D(a)b)_{st} = a_s (D(b))_t + (D(a))_s b_t =
\]

\[
a_s D_*(b_t) + D_*(a_s) b_t
\]

Hence \( D_* \) is a derivation on \( R_S \). ♦

**Remark**

We call the derivation \( D_* : R_S \rightarrow R_S \) as constructed above from the centrally-zero derivation \( D : R \rightarrow R \), the induced derivation on \( R_S \).

Next we give the second lemma, the importance of which for getting the results is not less than importance of the first one.

**Lemma 2:**

Let \( R \) be a ring for which \( Z(R) \) contains no proper zero divisors of \( R \) then:

1- If \( t \in Z(R) - \{0\} \) and \( r \in R \) such that \( tr = 0 \) then \( r = 0 \).

2- \( Z(R) - \{0\} \) is a multiplicative system in \( R \).

3- If \( R \) satisfies \( (CCP) \) then it is commutative.

4- \( (Z(R))_S = Z(R_S) \), for all multiplicative systems \( S \) in \( R \) with \( [S, R] = \{0\} \).

**Proof:**

1: Since \( 0 \neq t \in Z(R) \) so if \( r \neq 0 \) then \( t \) is a proper zero divisor of \( R \) which is a contradiction and hence \( r = 0 \).

2: Clearly \( 0 \notin Z(R) - \{0\} \) and if \( a, b \in Z(R) - \{0\} \) then \( a, b \in Z(R) \), and \( a \neq 0, b \neq 0 \) but since \( Z(R) \) is a subring of \( R \), so \( ab \in Z(R) \).
$Z(R)$ has no proper zero divisors we get $ab \neq 0$.
Hence $ab \in Z(R) - \{0\}$. Thus $Z(R) - \{0\}$ is a multiplicative system in $R$.

3: By (2) we have $Z(R) - \{0\}$ is a multiplicative system in $R$. Let $S = Z(R) - \{0\}$. Then clearly $[S, R] = \{0\}$ and hence by the (CCP) we get that $R_S$ is commutative. Now let $a, b \in R$ be any elements and since $S \neq \emptyset$ so there exists an $s \in S$. Then $a_s, b_s \in R_S$ and hence $a_s b_s = b_s a_s$
or $(ab)_{ss} = (ba)_{ss}$ and then $(ab, ss) \sim (ba, ss)$ so that $\exists t \in S$ such that
$t(abss - bass) = 0$ which implies $tss(ab - ba) = 0$. But then $t, s \in S$
implies $tss \in S$ so that $tss \neq 0$ and hence $tss \in Z(R) - \{0\}$. By applying (1) we get $ab - ba = 0$.
That is $ab = ba$, which means that $R$ is commutative.

4: It is clear that $Z(R)$ is a ring (in fact it is a subring of $R$) and since $[S, R] = \{0\}$ so $[S, Z(R)] = \{0\}$ and also $S \subseteq Z(R)$ which means that $S$
is a multiplicative system in the ring $Z(R)$ with $[S, Z(R)] = \{0\}$ which means talking about localization of $Z(R)$ at $S$ meaningfull.
Next we will show $(Z(R))_S \subseteq Z(R_S)$.

Let $a_s \in (Z(R))_S$, where $a \in Z(R)$, $s \in S$ then let $b_t \in R_S$ be any element,
where $b \in R, t \in S$. Since $a \in Z(R)$ and $b \in R$ so $ab = ba$ and then
$a_s b_t = (ab)_{st} = (ba)_{ts} = b_t a_s$ which means $[a_s, b_t] = 0$ for all $b_t \in R_S$.

Hence $[a_s, R_S] = \{0\}$ thus $a_s \in Z(R_S)$ and so that $(Z(R))_S \subseteq Z(R_S)$.

It remains to show $Z(R_S) \subseteq (Z(R))_S$ , so let $r_s \in Z(R_S)$ where $r \in R, s \in S$.
Now $\forall x \in R$ we have $x_s \in R_S$ and hence $r_s x_s = x_s r_s$ or
$(rx - xr)_{ss} = 0$ and so that $\exists u \in S$ such that $u(rx - xr) = 0$, where
$u \in S \subseteq Z(R) - \{0\}$ and hence by (1) we get $rx - xr = 0$ or $rx = xr$, thus
$r \in Z(R)$ and so that $r_s \in (Z(R))_S$ which gives
$Z(R_S) \subseteq (Z(R))_S$. Hence $Z(R)_S = Z(R_S)$.

Now it is the time for giving our main results.
**Theorem 1:**
Let \( R \) be a centrally prime ring in which \( Z(R) \) contains no proper zero divisors and \( D : R \rightarrow R \) is a centrally-zero derivation on \( R \) with \( D \neq 0 \). If \( D(x) \in Z(R), \forall x \in R \) then \( R \) is commutative.

**Proof:**
To show \( R \) satisfies \((CCP)\).So let \( S \) be any multiplicative system in \( R \) such that
\[ [S, R] = \{0\} \].Consider the induced derivation \( D_* : R_S \rightarrow R_S \) on \( R_S \),
where \( D_*(r_S) = (D(r))_S, \forall r_S \in R_S \).

Let \( D_* = 0 \). Now fix \( s \in S \) (since \( S \neq \phi \)) and let \( r \in R \) then
\[ r_S \in R_S \]. Hence \( (D(r))_S = D_*(r_S) = 0 \). Then there exists \( t \in S \) such that
\[ tD(r) = 0 \] but \( 0 \neq t \in S \subseteq Z(R) \). Hence \( t \in Z(R) \) \( \setminus \{0\} \). So by lemma \((2)\), \( D(r) = 0 \). Hence this result is true for all \( r \in R \) and thus \( D = 0 \) which is a contradiction. So \( D_* \neq 0 \). Now if \( x_S \in R_S \) where \( x \in R \) and \( s \in S \) then we have \( D_*(x_S) = (D(x))_S \in (Z(R))_S = Z(R_S) \).

Hence \( R_S \) is a prime ring and \( D_* : R_S \rightarrow R_S \) is a nonzero derivation on \( R_S \) such that \( D_*(x_S) \in Z(R_S), \forall x_S \in R_S \). Hence by theorem \((A)\), \( R_S \) is commutative.
So \( R \) satisfies \((CCP)\). Since \( Z(R) \) has no proper zero divisors we get \( R \) is commutative. 

**Theorem 2:**
Let \( R \) be a centrally prime ring with \( Z(R) \) has no proper zero divisors and \( D : R \rightarrow R \) be a centrally-zero derivation on \( R \) with \( D \neq 0 \). If \( U \) is a nonzero ideal of \( R \) such that \( D(u)u = uD(u), \forall u \in U \) then \( R \) is commutative.

**Proof:**
To show that \( R \) satisfies \((CCP)\). So let \( S \) be any multiplicative system in \( R \) such that \([S, R] = \{0\}\). To show the ideal \( U_S \) of \( R_S \) is nonzero. If \( U_S = 0 \), then let \( u \) be any element in \( U \) and fix \( s \in S \) (this is possible because \( S \neq \phi \)) so that \( u_S \in U_S \) and so \( u_S = 0 \), hence \( \exists t \in S \) such that
tu = 0, where 0 ≠ t ∈ S ⊆ Z(R), or t ∈ Z(R) − {0}. Hence u = 0 by lemma (2) and this, in consequence implies that U = 0 which is a contradiction and thus US ≠ 0.

Next let D∗ : RS → RS be the induced derivation on RS, where D∗(rs) = (D(r))s, ∀rs ∈ RS. To show D∗ ≠ 0. If D∗ = 0 then fix s ∈ S, so ∀r ∈ R we have rs ∈ RS. Hence (D(r))s = D∗(rs) = 0. Thus ∃t ∈ S such that tD(r) = 0, where 0 ≠ t ∈ S ⊆ Z(R) which means that t ∈ Z(R) − {0}. Hence by lemma (2) we get D(r) = 0 and this is true ∀r ∈ R. Hence D = 0 which is a contradiction, thus D∗ ≠ 0. Next ∀us ∈ US we have D∗(us)us = (D(u))sus = (uD(u))ss = uss(D(u))s = usD∗(us).

Thus we have RS is a prime ring (since R is centrally prime), US is a nonzero ideal of RS and D∗ is a nonzero derivation on RS such that D∗(us)us = usD∗(us), ∀us ∈ US. Hence by theorem (B) we get that RS is commutative. Hence R satisfies (CCP) and Z(R) being without proper zero divisors R is commutative.♦

**Theorem 3:**

Let R be a centrally prime ring in which Z(R) has no proper zero divisors and I a non zero ideal of R. Suppose that D : R → R is a centrally-zero derivation on R and n is a positive integer such that Dn(I) ⊆ Z(R) and Dn ≠ 0 then R is commutative.

**Proof:**

If R does not satisfy (CCP) then there exists a multiplicative system S with [S, R] = {0} for which R is not commutative. Note that [S, R] = {0} means that S ⊆ Z(R). Now suppose that IS = 0.

Since I ≠ 0 and S ≠ ∅ so ∃0 ≠ x ∈ I and s ∈ S. Then xs ∈ IS, and hence xs = 0 then ∃t ∈ S such that tx = 0, where 0 ≠ t ∈ S ⊆ Z(R).

Since x ≠ 0 so t is a proper zero divisor of R. Thus Z(R) contains a proper zero divisor of R which is a contradiction. Hence we get that IS ≠ 0.
Next let \( D_*, R_S \rightarrow R_S \) be the induced derivation on \( R_S \) (of lemma 1), where \( D_*(r_s) = ((D(r))_s), \forall r_s \in R_S \).

To show \( D^n_*(I_S) \subseteq Z(R_S) \). Let \( \lambda \in D^n_*(I_S) \). Then there exists \( r_s \in I_S \), where \( r \in I, s \in S \), such that \( \lambda = D^n_*(r_s) \). But \( r \in I \) implies \( D^n(r) \in D^n(I) \subseteq Z(R) \). Thus \( \lambda = D^n_*(r_s) = (D^n(r))_s \subseteq (Z(R))_S = Z(R_S) \).

Hence we get \( D^n_*(I_S) \subseteq Z(R_S) \).

Now \( R_S \) is a prime ring, \( I_S \) is a non-zero ideal of \( R_S \) and \( D_* \) is a derivation on \( R_S \) such that \( D^n_*(I_S) \subseteq Z(R_S) \) so by theorem (C) we get either \( D^n_*(I_S) = 0 \) or \( R_S \) is commutative and \( R_S \) being noncommutative so we get \( D^n_*(I_S) = 0 \).

If \( r \in R \) is any element then fix \( s \in S \) (since \( S \neq \phi \)) and thus \( r_s \in R_S \) then \( (D^n(r))_s = D^n_*(r_s) = 0 \) which means \( \exists t \in S \) such that \( t(D^n(r)) = 0 \), where \( 0 \neq t \in S \subseteq Z(R) \), and thus \( t \in Z(R) - \{0\} \). Hence by lemma (2), \( D^n(r) = 0 \) and this result is true \( \forall r \in R \) so that \( D^n = 0 \) which is a contradiction and hence \( R \) must satisfy (CCP) and since \( Z(R) \) contains no proper zero divisor we get that \( R \) is commutative (Lemma 2).

As a corollary to this theorem we give:

**Corollary 1:**

Let \( R \) be a centrally prime ring with \( Z(R) \) contains no proper zero divisors and \( D : R \rightarrow R \) is a centrally-zero derivation on \( R \). If \( n \) is a positive integer such that \( D^n(R) \subseteq Z(R) \) and \( D^n \neq 0 \) then \( R \) is commutative.

**Proof:**

Putting \( I = R \) in theorem (3) the result will follows.

**Theorem 4:**

Let \( R \) be a centrally prime ring with \( Z(R) \) contains no proper zero divisors and \( D : R \rightarrow R \) is a centrally-zero derivation on \( R \). If there exists a nonzero ideal \( J \) of \( R \) such that either

\[
xy + D(xy) = yx + D(yx), \forall x, y \in J
\]

or

\[
xy - D(xy) = yx - D(yx), \forall x, y \in J
\]

then \( R \) is commutative.
\textbf{Proof:}

If possible suppose that \( R \) does not satisfy (CCP), so there exists a multiplicative system \( S \) in \( R \) with \([S, R] = \{0\}\) but \( R_S \) is not commutative.

Then \( R_S \) is a prime ring (since \( R \) is centrally prime). Now let \( D_*: R_S \rightarrow R_S \) be the induced derivation on \( R_S \) (of Lemma 1) that is \( D_*(r_s) = (D(r))_s, \forall r_s \in R_S \).

To show that the ideal \( J_S \) is nonzero. Let \( J_S = 0 \). Now if \( x \in J \) then since \( S \neq \phi \) so \( \exists s \in S \). Hence \( x_s \in J_S \), which implies that \( x_s = 0 \) so \( \exists t \in S \) such that \( tx = 0 \), where \( 0 \neq t \in S \subseteq Z(R) \), which means \( t \in Z(R) - \{0\} \). Hence \( x = 0 \) (by lemma 2) which implies \( J = 0 \) and this is a contradiction and thus \( J_S \neq 0 \).

We take the first case which is \( xy + D(xy) = yx + D(yx), \forall x, y \in J \).

Let \( x_s, y_t \in J_S \) for \( x, y \in J \) and \( s, t \in S \).

Then \( x_s y_t + D_*(x_s y_t) = (xy)_{st} + D_*((xy)_{st}) = (xy)_{st} + (D(xy))_{st} = (xy + D(xy))_{st} = (yx + D(yx))_{ts} + (D(yx))_{ts} = y_t x_s + D_*(y(x)_{ts}) = y_t x_s + D_*(y_t x_s) \). That means \( J_S \) is a nonzero ideal of the prime ring \( R_S \) and \( D_*: R_S \rightarrow R_S \) is a derivation on \( R_S \) such that \( x_s y_t + D_*(x_s y_t) = y_t x_s + D_*(y_t x_s), \forall x_s, y_t \in J_S \). Hence from theorem (D) we get that \( R_S \) is commutative which is a contradiction. Hence \( R \) must satisfy (CCP). And since \( Z(R) \) has no proper zero divisors so by lemma 2, we get \( R \) is commutative.

And if we take the second case, that is \( x_s y_t - D_*(x_s y_t) = y_t x_s - D_*(y_t x_s), \forall x_s, y_t \in J_S \) then by the same technique we get that \( R \) is again commutative.

As a corollary to this theorem we give:

\textbf{Corollary 2:}

Let \( R \) be a centrally prime ring in which \( Z(R) \) has no proper zero divisors and \( D: R \rightarrow R \) is a centrally-zero derivation on \( R \).

If \( xy + D(xy) = yx + D(yx), \forall x, y \in R \) or \( xy - D(xy) = yx - D(yx), \forall x, y \in R \) then \( R \) is commutative.

\textbf{Proof:} Taking \( J = R \) in the theorem (4) we get the result.
Theorem 5: Let \( R \) be a centrally prime ring in which \( Z(R) \) has no proper zero divisors and \( U \) a nonzero left ideal of \( R \). Suppose that \( D: R \to R \) and \( G: R \to R \) are two centrally-zero derivations on \( R \) with \( D \neq 0 \) and such that \( D(u)u - uG(u) \in Z(R), \forall u \in U \), then \( R \) is commutative.

Proof: To show \( R \) satisfies \((CCP)\), let \( S \) be any multiplicative system in \( R \) with \([S, R] = \{0\}\). Clearly \( R_S \) is a prime ring. Consider the induced derivations \( D_*, G_*: R_S \to R_S \) on \( R_S \). To show \( U_S \) is nonzero left ideal of \( R_S \), let \( u_s, v_t \in U_S \), where \( u, v \in U, s, t \in S \). Clearly \( 0 \in U_S \) so \( U_S \) is a non-empty subset of \( R_S \). Now \( u_s - v_t = (ut - vs)s \in U_S \) (since \( ut - vs \in U, st \in S \) ), and if \( r_x \in R_S \) then \( r_xu_s = (ru)s \in U_S \) (since \( ru \in U \) and \( xs \in S \)). Hence \( U_S \) is a left ideal of \( R_S \). To show \( U_S \neq 0 \) and \( D_* \neq 0 \). If \( U_S = 0 \) then fix \( s \in S \) and now if \( u \in U \) then \( u_s = 0 \) and thus \( \exists t \in S \) such that \( tu = 0 \), where \( 0 \neq t \in S \subseteq Z(R) \), that is \( t \in Z(R) - \{0\} \) and hence by lemma (2), \( u = 0 \) which means \( U = 0 \) and this is a contradiction and thus \( U_S \neq 0 \) and if \( D_* = 0 \) then for any \( r \in R \) we have \( r_s \in R_S \). Hence \( (D(r))_S = D_*(r_s) = 0 \). So \( \exists x \in S \) such that \( xd(r) = 0 \), where \( 0 \neq x \in S \subseteq Z(R) \) or \( x \in Z(R) - \{0\} \). Thus by lemma (2), \( D(r) = 0 \) and this is true \( \forall r \in R \) so \( D = 0 \) which is a contradiction and thus \( D_* \neq 0 \). Next \( \forall u_s \in U_S \) we have \( D_*(u_s)u_s - u_sG_*(u_s) = (D(u))Su_s - u_s(G(u))_S = (D(u)u)_{SS} - (uD(u))_{SS} = (D(u)u - uD(u))_{SS} \in Z(R_S) \). Thus \( R_S \) is a prime ring, \( U_S \) is a nonzero left ideal of \( R_S \) and \( D_*, G_*: R_S \to R_S \) are derivations on \( R_S \) with \( D_* \neq 0 \) and such that \( D_*(u_s)u_s - u_sG_*(u_s) \in Z(R_S), \forall u_s \in U_S \) and hence by theorem (E), \( R_S \) is commutative so \( R \) satisfies \((CCP)\) and since \( Z(R) \) is without proper zero divisors so \( R \) is commutative.\( \blacksquare \)
Corollary 3:
Let $R$ be a centrally prime ring in which $Z(R)$ is without proper zero divisors and $U$ is a nonzero left ideal of $R$. If $D : R \rightarrow R$ is a centrally-zero derivation on $R$ with $D \neq 0$ such that $D(u)u + uD(u) \in Z(R), \forall u \in U$, then $R$ is commutative.

Proof:
Define $G : R \rightarrow R$ by $G(r) = -D(r), \forall r \in R$. To show $G$ is a derivation on $R$. First if $a = b \in R$ then $D(a) = D(b)$ or $-D(a) = -D(b)$ and hence $G(a) = G(b)$ So $G$ is a mapping.

Now for $a, b \in R$ we have $G(a + b) = -D(a + b) = -(D(a) + D(b)) = -D(a) + (-D(b)) = G(a) + G(b)$ and also $G(ab) = -D(ab) = -(aD(b) + D(ab)) = a(-D(b)) + (-D(a))b = aG(b) + G(a)b$ and thus $G$ is a derivation on $R$. Now if $S$ is any multiplicative system in $R$ with $[S, R] = \{0\}$ then $\forall s \in S$ we have $G(s) = -D(s) = 0$. Hence $G$ is also a centrally-zero derivation. Next $\forall u \in U$ we have

$$D(u)u - uG(u) = D(u)u - u(-D(u)) = D(u)u + uD(u) \in Z(R).$$

Thus $R$ is a centrally prime ring in which $Z(R)$ has no proper zero divisors, $U$ is a nonzero left ideal of $R$ and $D, G : R \rightarrow R$ are centrally-zero derivations with $D \neq 0$ such that $D(u)u - uG(u) \in Z(R), \forall u \in U$.

Hence by the theorem (5) we get $R$ is commutative.

Theorem 6:
Let $R$ be a centrally prime ring with $Z(R)$ has no proper zero divisors and $U$ is a nonzero left ideal of $R$. If $D : R \rightarrow R$ is a centrally-zero derivation on $R$ with $D \neq 0$ such that $D(U) \subseteq Z(R)$, then $R$ is commutative.

Proof:
To show $R$ satisfies (CCP), let $S$ be any multiplicative system in $R$ with $[S, R] = \{0\}$. Clearly $R_S$ is a prime ring (since $R$ is centrally prime).

Now consider the induced derivation $D_* : R_S \rightarrow R_S$ on $R_S$.

As we have done in theorem (5) we can show $U_S$ is a left ideal of $R_S$ and $U_S \neq 0, D_* \neq 0$ and $D_*(U_S) \subseteq Z(R_S)$. Hence $R_S$ is a prime ring, $U_S$ is a nonzero left ideal of $R_S$ and $D_* : R_S \rightarrow R_S$ is a nonzero derivation on $R_S$ such that $D_*(U_S) \subseteq Z(R_S)$. Hence from theorem (E)
we get $R_S$ is commutative and hence $R$ must satisfy (CCP) and $Z(R)$ being without proper zero divisors, therefore $R$ becomes commutative. ♦

**Theorem 7:**
Let $R$ be a centrally prime ring in which $Z(R)$ has no proper zero divisors and $D : R \to R$ is a centrally-zero derivation on $R$ with $D \neq 0$. If $I$ is a nonzero ideal of $R$ such that $D([x, y]) = [x, y], \forall x, y \in I$ then $R$ is commutative.

**Proof:**
If $R$ does not satisfy (CCP) then there exists a multiplicative system $S$ in $R$ with $[S, R] = \{0\}$ but $R_S$ is not commutative. Then let $D_* : R_S \to R_S$ be the induced derivation on $R_S$ where $D_*(r_s) = (D(r))_S, \forall r_s \in R_S$.

Since $R$ is centrally prime so $R_S$ is a prime ring.

We will show $I_S \neq 0$ and $D_* \neq 0$. If $I_S = 0$, then $S \neq \phi \Rightarrow \exists s \in S$ so for any $x \in I$ we have $x_s = 0$ which implies that $\exists t \in S$ such that $tx = 0$, where $0 \neq t \in S \subseteq Z(R).$ So $t \in Z(R) - \{0\}$. Thus by lemma (2), $x = 0$.

Hence $I = 0$, which is again a contradiction and so $I_S \neq 0$.

Now suppose that $D_* = 0$ then fix $s \in S$ (since $S \neq \phi$), now if $r \in R$ is any element then we have $r_s \in R_s$. Hence $(D(r))_S = D_*(r_s) = 0$.

Thus $\exists u \in S$ such that $uD(r) = 0$, where $0 \neq u \in S \subseteq Z(R)$, that is $u \in Z(R) - \{0\}$. Hence by lemma (2), we get $D(r) = 0$ and this result is true for all $r \in R$ which means that $D = 0$ and this contradicts the fact that $D$ is a nonzero derivation on $R$. Hence $D_* \neq 0$.

Next let $a_s, b_t \in I_S$, where $a, b \in I, s, t \in S$. Then

$$D_*([a_s, b_t]) = D_* (a_s b_t - b_t a_s) = D_* ((ab)_{st} - (ba)_{ts}) =$$
$$D_* ((ab - ba)_{st}) = (D(ab - ba))_{st} =$$
$$((a, b))_{st} = (ab - ba)_{st}$$

$$= (ab)_{st} - (ba)_{ts} = a_s b_t - b_t a_s = [a_s, b_t].$$

Hence $R_S$ is a prime ring, $I_S$ is a nonzero ideal of $R_S$ and $D_*$ is a nonzero
derivation on \( R_S \) such that \( D_*([a_s, b_t]) = [a_s, b_t], \forall a_s, b_t \in I_S \).

Hence from theorem (F) we get that \( R_S \) is commutative which is a contradiction thus \( R \) must satisfy (CCP) and \( Z(R) \) being without proper zero divisors we get \( R \) is commutative. ♦

**Corollary 4:**
Let \( R \) be a centrally prime ring in which \( Z(R) \) has no proper zero divisors and \( D : R \rightarrow R \) be a centrally-zero derivation on \( R \) with \( D \neq 0 \) such that \( D([x, y]) = [x, y], \forall x, y \in R \) then \( R \) is commutative.

**Proof:**
Putting \( I = R \) in the theorem (7), the result will follows. ♦

**References**


الحلقات الأولية مركزيا والتي تكون تبادلية

عادل قادر جبار
كلية العلوم- جامعة السليمانية

الخلاصة

في هذا البحث قدمنا تعريف الحلقات الأولية مركزيا حيث أن هدفنا الرئيسي هو تصنيف الحلقات الأولية مركزيا والتي تكون تبادلية وقد تمكنا من إعطاء شروط عديدة والذي يجعل كل واحد منها من الحلقة الأولية مركزيا حلقة تبادلية.