Strongly Pure Ideals And Strongly Pure Sub-modules

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ABSTRACT

Let $R$ be a ring with unity, and let $M$ be an unitary $R$-module. In this work we present strongly pure ideal (sub module) concept as a generalization of pure ideal (sub module). Also we generalize some properties of strongly pure ideal (sub module). And we study strongly regular ring (R-module).

Keywords: pure ideal, strongly pure ideal, local ring, pure sub-module, strongly pure sub module, flat module, superfluous sub module.
الممخص

مثاليات النقية بقوة والمقاسات الجزئية النقية بقوة

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1. INTRODUCTION

Let R be a ring with unity, and let M be a unitary R-module. Let L and K be sub modules of M, the residual of K by L, denoted by [K:L] is the set of all x in R such that xL ⊆ K. The annihilator of M, denoted by ann M, is [0:M]. For each m ∈ M, the annihilator of m, denoted by ann(m), is [0:Rm]. Let M be a multiplication R-module and N is a sub module of M. Then N=IM for some ideal I of R. Note that I ⊆ [N:M] and hence N=IM ⊆ [N:M]M ⊆ N, so that N=[N:M]M. Also K is a multiplication sub module of M if and only if N∩K=[N:K]K, for every sub module N of M, [24]. Finitely generated faithful multiplication modules are cancellation modules, [29]. We introduce the concept of idempotent sub module as follows.
A sub module $N$ of $M$ is an idempotent if and only if $N = [N:M]N$. If $M$ is a finitely generated faithful multiplication $R$-module, then $N$ is an idempotent sub module of $M$ if and only if $[N:M]$ is an idempotent ideal of $R$. Brown-McCoy (1950) in [9] define regular rings. And Fieldhouse (1969) generalize regular rings to regular modules in [16]. Ware (1971) in [30] and Zelmanowilz (1972) in [31] also study regular modules. E. Anderson and Fuller [5] called the sub module $N$ is a pure of $M$ if $IN = N \cap IM$ for every ideal $I$ of $R$. Although $R$. Ribenboim [27] define $N$ to be pure in $M$ if $rM \cap N = rN$ for each $r \in R$.

This work include four sections. In section one, we introduce strongly pure ideals concept as generalization of pure ideals. An ideal $I$ of $R$ is said to be strongly pure, if for each $x \in I$ there exists a prime element $p \in I$ such that $x = xp$. Pure ideal is deferent from strongly pure ideal, thus we give examples which indicate that two classes are different. However, we put some conditions under which the two classes are equivalent as we see in (Prop.2-4) that $R$ is factorial ring, such that each non zero and non unit element of $R$ is irreducible. Moreover, we give some properties of strongly pure ideals as the intersection of two ideals is strongly pure ideal if one of them is strongly pure ideal (Prop.2-7). In (Prop.2-9) prove that if the direct summand of two ideals is strongly pure, then one of them is strongly pure. If $I$ is strongly pure ideal of $R$, then $I_A$ is strongly pure ideal of $R_A$ (Prop.2-10).

Also, we present in section two strongly regular ring that a ring $R$ is called strongly regular if and only if for each $x \in R$, there exists prime element $p \in R$ such that $x = xp$. In section three, we introduce strongly pure sub module concept. A sub module $N$ of an $R$-module $M$ is called strongly pure, if there exists prime ideal $P$ of $R$, such that $N \cap Mp = Np$, for each $p \in P$. Pure submodule is deferent from strongly pure sub module, thus we give examples which indicate that two classes are different. However, we put some conditions under which the two classes are equivalent (Prop.4-3). And we study some properties of strongly pure sub modules. Also, we get the main result of strongly pure sub modules in (Th.4-13).

Finally in section four, we present strongly regular module concept as generalization of regular module. Also generalize characterization and some properties of regular module to strongly regular module.
2. STRONGLY PURE IDEALS

In this section, we introduce a generalization for pure ideal concept namely strongly pure ideal.

Note: From now on, strongly pure ideal means right strongly pure ideal unless otherwise stated. First we recall definitions of pure ideals and prime ideals.

Recall that an ideal \( I \) of a ring \( R \) is said to be right (left) pure if for each \( x \in I \), there exists \( y \in I \) such that \( x = x.y \) (\( x = y.x \)), \([2] \) & \([17]\).

Definition: An ideal \( I \) of a ring \( R \) is called strongly right (left) pure if for each \( x \in I \), there exists a prime element \( p \in I \) such that \( x = x.p \) (\( x = p.x \)).

Remarks and Examples:
1- Each strongly pure ideal is pure. But the converse is not true in general, as we see in the following example:

\[ (\overline{3}) = \{0, \overline{3}\} \] is a strongly pure ideal of a ring \( Z_6 \), since \( \overline{0} = \overline{0} \cdot \overline{3} \) and \( \overline{3} = \overline{3} \cdot \overline{3} \). But \( (\overline{2}) = \{0, \overline{2}, \overline{4}\} \) is not strongly pure ideal of a ring \( Z_6 \), since there is no prime element \( p \in (\overline{2}) \) such that \( \overline{2} = \overline{2} \cdot p \).

2- If \( I \) is strongly pure ideal of a ring \( R \), then \( JI = J \cap I \), for each ideal \( J \) of \( R \).

Recall that an ideal \( I \) of a ring \( R \) is called idempotent if \( I^2 = I \), \([20]\).

3- Each ideal generated by prime idempotent element is strongly pure ideal.

Proof: Let \( I = (p) \) be an ideal generated by prime element \( p \), such that \( p = p^2 \). If \( x \in I \), there exists \( r \in R \) such that \( x = rp \), implies \( x = rp = rp^2 = rpp = xp \). Therefore \( I \) is strongly pure ideal of a ring \( R \). In general

4- If \( I \) is an ideal of a ring \( R \) generated by a prime idempotent elements \( p_i \), where \( i = 1, 2, \ldots, n \). Then \( I \) is a strongly pure ideal of \( R \).

Proof: Let \( I \) be an ideal generated by prime idempotent elements \( p_i \), where \( i = 1, 2, \ldots, n \). And let \( x \in I \), there exists \( r_i \in R \) such that \( x = \sum_{i=1}^{n} r_i p_i = \sum_{i=1}^{n} p_i r_i = \sum_{i=1}^{n} r_i p_i \). Let \( p_i = 1 - \prod_{i=1}^{n} (1 - p_i) \), implies that

\[ \sum_{i=1}^{n} r_i p_i (1 - \prod_{i=1}^{n} (1 - p_i)) = \sum_{i=1}^{n} r_i p_i \prod_{i=1}^{n} (1 - p_i) \]

\[ (1 - p_i) = x - \prod_{i=1}^{n} (1 - p_i) (\sum_{i=1}^{n} r_i p_i) = x. \] Therefore \( I \) is strongly pure ideal of \( R \).
5- Each strongly pure ideal is idempotent.

**Proof**: Let I be a strongly pure ideal of R, and let \( x \in I \). Then there exists a prime element \( p \in I \), such that \( x = xp \). But \( x \in I_1 \), thus \( x \in I^2 \). Hence \( I \subseteq I^2 \), and it is clear that \( I^2 \), implies \( I = I^2 \). Therefore I is an idempotent ideal of R. Recall that an element \( a \in R \) is called irreducible if it is non zero non unit and whenever \( a = bc \) where \( b, c \in R \), then either \( b \) or \( c \) is a unit element of R, [18]. An integral domain R is a factorial ring if there is a set S of non zero non unit of R such that every non zero element of R can be written uniquely in the form \( ua_1a_2...a_k \), where \( u \) is a unit of R and \( a_1, a_2, ..., a_k \in S \). Except for the order in which the factors are written , [20]

The following lemma taken from [20].

**Lemma**: Let R be a factorial ring, and let S be a set whose existence in required in the definition. Then every irreducible element of R is prime, every element of S is prime, and every prime of R is the product of unit of R and an element of S. The following result gives some conditions to get strongly pure ideals from pure ideals.

**Proposition**: Let R be a factorial ring, and let I be an ideal of R, such that each non zero and non unit element of R is irreducible. Then I is strongly pure ideal if and only if I is pure ideal.

**Proof**: Let I be a pure ideal of R, and let \( x \in I \), there exists \( y \in I \) such that \( x = xy \). Since \( y \in R \) is irreducible element of R, so it is prime in I (Lemma2-3). Therefore I is strongly pure ideal of R.

The converse is clear. In general right strongly pure ideal is not left, and left strongly pure ideal is not right. As the following example.

**Example**: Let \( R = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{Z}_2 \) be the ring of 2×2 matrices over the field \( \mathbb{Z}_2 \). Then there are two ideals I and J namely I=\( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \), J=\( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \), \( \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \). Clearly I is right strongly pure ideal and J is not left strongly pure ideal. Recall that a ring R is said to be reduced if R contains no non zero nilpotent element, [22]. The following proposition gives condition to get right strongly pure ideal from left strongly pure ideal,
Proposition: Let R be a reduced ring, and let I be any ideal of R. Then I is a right strongly pure if and only if I is a left strongly pure.

Proof: Let I be a right strongly pure ideal of R and let \( x \in I \), there exists prime element \( p \in I \) such that \( x = xp \). Since R is a reduced ring, then \((x-xp)^2 = 0\), so \( x-px = 0 \), thus \( x = px \). Therefore I is a left strongly pure ideal of R. The converse is similar, hence it is omitted.

The following proposition gives some property of strongly pure ideals.

Proposition:
1- Let I and J are two ideals of a ring R. If I is strongly pure ideal of R, then \( I \cap J \) is a strongly pure ideal of R.
2- Let I and J are two ideals of a ring R such that \( I \subseteq J \). If \( I \cap J \) is strongly pure ideal of R, then I is strongly pure ideal of R.

Proof: Clear.

The following corollaries is immediate consequence of previous proposition.

Corollary: Let I and J are two strongly pure ideals of R, then \( I \cup J \) is also strongly pure ideal of R.

Corollary: Let I and J are two ideals of R, then I is strongly pure ideal if and only if \( I \cap J \) is strongly pure ideal of R. The following proposition gives another property of strongly pure ideals.

Proposition: Let I and J are two ideals of a ring R, if \( I \oplus J \) is strongly pure ideal of R. Then either I or J is strongly pure ideal of R.

Proof: Let \( x \in I \) and \( x \in J \), implies \( x+y \in I \oplus J \). Since \( I \oplus J \) is strongly pure ideal of R, there exists prime element \( p \in I \oplus J \), where \( p = p+0 \in I \oplus J \), such that \( x+y=(x+y)p=(x+y)(p+0)xp+yp \in I \oplus J \). Since \( yp \in I \cap J \), and \( I \cap J = \{0\} \), hence \( yp = 0 \). Thus \( x = xp \in I \). Therefore I is strongly pure ideal of R. And if \( p = 0 + p \in I \oplus J \), then by the same method we can get J is strongly pure ideal of R.

Next, from (prop. 2–4) and (prop. 2–10) we can get the following result.

Corollary: Let R be a factorial ring, and let I and J are two ideals of R, such that \( I \oplus J \) is strongly pure ideal of R. Then I and J are strongly pure ideals of R.
Proposition: If I is a strongly pure ideal of a ring $R$, then $I_A$ is strongly pure ideal of a ring $R_A$, for each maximal ideal $A$ of $R$.

Proof: Let $a/r \in I_A$, where $a \in I$ and $r \in R-A$. Since I strongly pure ideal of $R$, there exists prime element $p \in I$ such that $a = a.p$, implies that $a/r = a.p/r = a/r.p/1$, where $p/1$ is prime in $I_A$. Thus $I_A$ is strongly pure ideal of $R_A$.

Remark: Let $I$ be strongly pure ideal of $R$, such that $I \subseteq J(R)$, then $I = \{0\}$

Proof: Let $x \in I$, since $I$ is strongly pure ideal of $R$, there exists a prime element $p \in I$, such that $x = x.p$, implies $x(1-p) = 0$. And since $I \subseteq J(R)$, then $p \in J(R)$, hence $x = 0$, so $I = \{0\}$. Recall that a ring $R$ is called local ring if $R$ is commutative, with unique maximal ideal, [11].

Proposition: Let $I$ be an ideal of commutative ring $R$. Then $I$ is strongly pure ideal of $R$ if and only if $I_A = (0)$ or $I_A = R_A$ for each maximal ideal $A$ of $R$.

Proof: If $I$ is a strongly pure ideal of $R$, then $I_A$ is strongly pure ideal of $R_A$ (Prop.2–12). If $I_A \neq R_A$, implies $R_A$ is local ring, hence $I_A \subseteq J(R_A)$. Therefore $I_A = (0)$ (Remark 2–13).

3. STRONGLY REGULAR RINGS

Let $R$ be a commutative ring with non-zero identity. Before we state some results, we give important definition which will be used here and later. Recall that a ring $R$ is regular if and only if for each $x \in R$, there exists $y \in R$ such that $x = xyx$, [26].

Now, we present strongly regular ring concept as generalization of regular ring

Definition: A ring $R$ is called strongly regular if and only if for each $x \in R$, there exists a prime element $p \in R$ such that $x = xp$. Also: A ring $R$ is called strongly regular if each element of $R$ is strongly regular

Remarks and Examples:
1- $\mathbb{Z}_4$ is strongly regular ring, but $\mathbb{Z}_6$ is not strongly regular ring.
2- Each strongly regular ring is a regular ring. But the converse is not true in general, as we see in the following example.
$\mathbb{Z}_6$ is a regular ring, but it is not strongly regular ring. Recall that right duo ring if all whose right ideals are two sided ideals. A left duo ring is similarly defined. A duo ring is right and left duo ring. [10]. The following results follow from (Prop.2-4) and from [14, Prop.2-24].
3- Let R be right duo factorial ring, such that each ideal of R is irreducible. Then R is strongly regular if and only if each ideal of R is left strongly pure.

4- Let R be a factorial ring, such that each ideal of R is irreducible. Then R is strongly regular ring if and only if each ideal of R is strongly pure.

In this section we give characterizations and some useful properties of strongly regular ring.

**Proposition**: Let R be a factorial commutative ring, such that each ideal of R is irreducible. Then R is strongly regular ring if and only if \( R_A \) is field, for each maximal ideal A of R.

**Proof**: Let \( R_A \) be a field, for each maximal ideal A of R, and let \( I \) be an ideal of R. Then \( I_A \) is an ideal of \( R_A \), since \( R_A \) is field, then \( R_A \) has no proper ideal. Thus either \( I_A = (0) \) or \( I_A = R_A \) \([14]\). Implies that I is strongly pure ideal of R (Prop.2-15). Therefore R is strongly regular ring, (Remark 3-3(5)).

Conversely, let J be an ideal of R, there exists an ideal I of R such that \( J = I_A \). But R is strongly regular ring, thus I is strongly pure ideal of R. Hence either \( I_A = (0) \) or \( I_A = R_A \) (Prop.2-15), its mean that \( R_A \) has no proper ideal. Therefore \( R_A \) is field.

We end this section by the following proposition.

**Proposition**: Let \( R_1 \) and \( R_2 \) are two rings, such that \( R_1 \oplus R_2 \) is strongly regular ring. Then either \( R_1 \) or \( R_2 \) is strongly regular ring.

**Proof**: Let \( r_1 \in R_1 \) and \( r_2 \in R_2 \), implies \( r_1 + r_2 \in R_1 \oplus R_2 \). Put \( x = r_1 + r_2 \), since \( R_1 \oplus R_2 \) is strongly regular ring, there exists prime element \( p = p + 0 \in R_1 \oplus R_2 \), such that \( x = px = (r_1 + r_2) p \) where \( r_1 + r_2 = r_1 p + r_1 p + r_2 p + r_2 p \), But \( r_1 p + r_2 p \) are elements of \( R_1 \cap R_2\), and \( R_1 \cap R_2 \) is (0). Thus \( x = r_1 + r_2 = r_1 p + r_2 p \), implies that \( r_1 = r_1 p \in R_1 \). Therefore \( R_1 \) is strongly regular ring. And if \( 0 + p \in R_1 \oplus R_2 \), by the same method we can get \( r_2 = r_2 p \in R_2 \), hence \( R_2 \) is strongly regular ring.

**4. STRONGLY PURE SUB MODULES**

Let M be a unitary R-module, and let R be a commutative ring with unity, such that each ideal of R is a proper ideal. Recall that a sub module N of an R-module M is called pure if \( N \cap Mr = Nr \) for each \( r \in R \). Also, we define N is pure sub module if \( N \cap MI = NI \), for each ideal I of a ring R. Now, we start this section by the following definition.

**Definition**: A sub module N of an R-module M is called strongly pure, if there exists prime
ideal P of a ring R, such that N∩Mp=Np, for each p∈P. Also; we define N is strongly pure sub module if there exists a prime ideal P of a ring R, such that N∩MP=NP.

Remarks and Examples:

1- (3) is a strongly pure sub module of Z_{6}-module. But (2) is not strongly pure sub module of Z_{6}-module.

2- Each strongly pure sub module is pure. But the converse is not true in general, as we see in the following example (3) is pure sub module of Z_{12}-module. But it is not strongly pure sub module of Z_{12}-module. Next, we study the relation between pure sub module and strongly pure sub module by the following proposition.

Proposition: Let M be an R-module, and R/I is an integral domain, for each ideal I of R. Then N is strongly pure sub module of M if and only if N is pure sub module of M.

Proof: Let N be a pure sub module of M, thus N∩MI=NI for each ideal I of R. Since R/I is an integral domain, hence I is a prime ideal, [21]. Therefore N is strongly pure sub module of M. The converse is clear. The following remark gives characterization of strongly pure sub module.

Remark: Let M be an R-module, and let R/I be an integral domain, for each ideal I of R. If N is a sub module of M, then the following statements are equivalent:

1- N is strongly pure sub module.

2- N ∩ MI = NI for each ideal I of a ring R.

3- N ∩ MI = NI for each finitely generated ideal I of R. [14, Prop.(2-8)].

The following propositions show some properties of strongly pure sub modules.

Proposition: Let N be a sub module of an R-module M, and let L be a sub module of N. If N is pure sub module of M, and L is strongly pure sub module of N, then L is strongly pure sub module of M.

Proof: Since L is strongly pure sub module of N, then there exists a prime ideal P of R, such that L∩NP=LP. And since N is pure sub module of M, then N∩MP=NP, implies that L∩N∩MP=LP. And since L⊆N, then L∩N=L, implies L∩MP=LP. Therefore L is strongly pure sub module of M. The following corollary follow from the previous proposition.

Corollary: Let N and L are sub modules of an R-module M. If N is a pure sub module of M, and N∩L is strongly pure sub module of N. Then N∩L is strongly pure sub module of M.
**Proposition:** Let $M$ be an $R$-module and let $R/I$ be an integral domain for each ideal $I$ of $R$. Then every direct summand sub module of $M$ is strongly pure sub module of $M$.

**Proof:** Suppose $N$ is a direct summand of an $R$-module $M$, then $M = N \oplus L$ for some sub module $L$ of $M$. As $NI \subseteq N \cap MI$ is trivial, where $I$ is an ideal of $R$. We want to prove the reverse inclusion. Let $x \in N \cap MI$, implies $x \in MI = (N \oplus L)I$, then there exists $a_i \in N$, $b_i \in L$ and $r_i \in I$ such that $x = \sum_{i=1}^{n} (a_i + b_i) r_i$. Then $x = \sum_{i=1}^{n} a_i r_i + \sum_{i=1}^{n} b_i r_i$, so $x = \sum_{i=1}^{n} a_i r_i = \sum_{i=1}^{n} b_i r_i$. But $x \in N$, thus $x = \sum_{i=1}^{n} a_i r_i \in N$. Hence $\sum_{i=1}^{n} b_i r_i \in N \cap L$. Since $N \cap L = (0)$, thus $\sum_{i=1}^{n} b_i r_i = 0$, and $x = \sum_{i=1}^{n} a_i r_i = 0$. It is mean $x = \sum_{i=1}^{n} a_i r_i$, so $x \in NI$. Thus $N \cap MI \subseteq NI$. Therefore $N$ is pure sub module of $M$. And since $R/I$ is an integral domain. Thus $N$ is strongly pure sub module of $M$ (Prop. 4-3). Before, we give the main theorem of pure sub modules, we will need the following well-known lemmas. Recall that an $R$-module $M$ is said to be flat if given any exact sequence $0 \rightarrow A \rightarrow B$ of right $R$-modules, the sequence $0 \rightarrow A \otimes_R M \rightarrow B \otimes_R M$ is exact.[7].

**Lemma :**[23] If $M$ is a flat $R$-module, then for any sub module $N$ of $M$, the following statements are equivalent.

1- $M/N$ is flat.
2- $N \cap MI = NI$ for each ideal $I$ of $R$.
3- $N \cap MI = NI$ for each finitely generated ideal $I$ of $R$.

Recall that a sub module $N$ of an $R$-module $M$ is called superfluous if and only if $N + L \neq M$, for each proper sub module $L$ of $M$, [19].

**Lemma :** (Nakayama’s lemma) [5], [19] and [28].

For a left ideal $I$ of a ring $R$, the following statements are equivalent:

1- For every finitely generated $R$-module $M$, if $MI = M$, then $M = 0$.
2- For every finitely generated $R$-module $M$, $MI$ is superfluous in $M$.

Recall that a right $R$-module $M$ is said to be a multiplication $R$-module if for each sub module $N$ of $M$, there exists an ideal $I$ of a ring $R$, such that $N = MI$, [1], [8] and [15]. And an $R$-module $M$ is called faithful if $\text{ann} M = (0)$, [15].
Lemma: [23] Let $N$ be a sub module of finitely generated faithful multiplication $R$-module $M$. If $N=[N:M]N$, then $N$ is an idempotent sub module of $M$ if and only if $[N:M]$ is an idempotent ideal of a ring $R$. The following theorem gives several characterization of strongly pure sub modules of faithful multiplication $R$-module with strongly pure annihilator.

Theorem: Let $R/I$ be an integral domain and let $M$ be a multiplication $R$-module , with strongly pure annihilator. If $N$ is sub module of $M$. Then statements (1) to (9) are equivalent, and further if $M$ is finitely generated and faithful, then statements (1) to (10) are equivalent.

1. $N$ is a strongly pure sub module of $M$.
2. $N$ is a multiplication and idempotent sub module of $M$.
5. $Rx=[N:M]x$ for each $x \in N$.
6. $R=[N:M]+\text{ann}(x)$ for each $x \in N$.
7. $R=\sum_{n \in N} [Rn:M]+\text{ann}(x)$ for each $x \in N$.
8. For each $x \in N$, there exists $a \in [N:M]$, such that $x=x a$.
9. For each maximal ideal $P$ of $R$ either $N_P=0_P$ or $N_P=M_P$.

Proof: (1)⇒(2) Let $K$ be a sub module of $M$, then $K=[K:M]M$. Since $N$ is a strongly pure sub module of $M$ and $R/I$ is an integral domain for each ideal I of R, thus $N$ is pure sub module of $M$ (Prop.4-3), we infer that $[K:N]N=N \cap [K:N]M \supseteq N \cap [K:M]M=N \cap [K:N]N$, and $N$ is a multiplication. Since $N$ is pure sub module of $M$, we have that $[N:M]N=[N:M]N \cap N=N$, and hence $N$ is idempotent sub module of $M$.


(4)⇒(2) Take $K=N$.

(3)⇒(7) Let $x \in N$, then $Rx=[N:M]x$. Since $N$ is multiplication, it follows by [3, Lemma 1-1(iv)] that $Rx=[N:M]x=(\sum_{n \in N} [Rn:M])x$, and then...
[29 ,Corollary to Theorem 9] gives that \( R = \sum_{x \in N} [Rn:M] + \text{ann}(x) \).

(5)\( \Leftrightarrow \) (8) Clear.

(5)\( \Rightarrow \) (9) Let \( P \) be any maximal ideal of \( R \). We discuss two cases.

**Case 1:** If \([N:M] \subseteq P \). Then for each \( x \in N \), \((Rx)_P = [N:M]_P(Rx)_P \subseteq P(Rx)_P \subseteq (Rx)_P\), so \((Rx)_P = P(Rx)_P\). By Nakayama’s Lemma, \((Rx)_P = 0\). Hence \( N_P = 0 \).

**Case 2:** If \([N:M] \not\subseteq P \), there exists \( p \in P \) such that \( 1 - p \in [N:M] \), and hence \( (1 - p)M \subseteq N \). It follows that \( N_P = M_P \).

(9)\( \Rightarrow \) (1) If \( N_P = 0 \) or \( N_P = M_P \) for every maximal ideal \( P \) of \( R \), then for every ideal \( I \) of \( R \), \( NI = N \cap MI \) is true locally, [21]. Thus \( N \) is pure submodule of \( M \), and since \( R/I \) is an integral domain for each ideal \( I \) of \( R \), thus \( N \) is strongly pure submodule of \( M \) (Prop.4-3).

(1)\( \Rightarrow \) (10) Let \( M \) be a finitely generated faithful multiplication \( R \)-module and let \( I \) be any ideal of \( R \).

Then \( NI = N \cap MI \). Hence \([NI:M] = [N\cap MI:M] = [N:M]\cap[M:MI:M] = [N:M]\cap I\), [18]. We need to show that \([NI:M] = [N:M]I\). Obviously, \([N:M]I \subseteq [NI:M]\). Conversely, let \( x \in [NI:M]\).

Then \( xM \subseteq NI = ([N:M]I)M \). But \( M \) is cancellation, thus \( x \in [N:M]I \), and hence \([NI:M] \subseteq ([N:M]I)M\).

(10)\( \Rightarrow \) (2) For this part it is not necessary to assume \( M \) is finitely generated or faithful.

Assume \([N:M]I = [N:M] \cap I\) for all ideals \( I \) of \( R \). Take \( I = [N:M] \). Then \([N:M]^2 = [N:M]\), and hence \([N:M]\) is an idempotent ideal of \( R \). It follows that \( N \) is idempotent submodule of \( M \) (Lemma 4-10).

Now, to prove that \( N \) is multiplication, let \( K \) be any submodule of \( M \), and let \( I = [K:M] \).

Then \((K \cap N):M = [K:M] \cap [N:M] = [K:M][N:M] \subseteq [K:N][N:M]\), and hence \( K \cap N = [K \cap N]:M \subseteq [K:N][N:M] \subseteq [K:N]N \subseteq K \cap N\), so that \( K \cap N = [K:N]N\), and \( N \) is multiplication. Next, we get the following result.

**Corollary:** Let \( R/I \) be an integral domain, and let \( M \) be a finitely generated faithful multiplication \( R \)-module. If \( N \) is strongly pure submodule of \( M \), then the following statements hold:

1- \( N = [N:M]N \).
2- \( \text{ann}N = \text{ann}[N:M] \).
3- \([N:M]\) is strongly pure ideal of \(R\).

Proof: (1) follows from Theorem 4-11.

(2) As \(N\) is strongly pure submodule of \(M\), we have \(NP=N\cap MP\), for some prime ideal \(P\) of \(R\). Taking \(P=\text{ann}N\), we get \(0=N\cap(\text{ann}N)M\), and hence \(0=[0:M]=((N:([\text{ann}N]M):M]=[N:M]\cap([\text{ann}N]M:M)]=[N:M]\cap\text{ann}N=[N:M]\text{ann}N\).

Hence \(\text{ann}N\subseteq\text{ann}[N:M]\).

Conversely, if \(x\in\text{ann}[N:M]\), then \(x[N:M]=0\), and hence \(xN=x[N:M]N=0\), so that \(x\in\text{ann}N\) and \(\text{ann}[N:M]\subseteq\text{ann}N\). Therefore \(\text{ann}N=\text{ann}[N:M]\).

(3) Let \(a\in[N:M]\), implies \(aM\subseteq N\), and by theorem(4-11,(3)) \(aM=[N:M]aM\), and hence \(Ra=[N:M]a\). Therefore \([N:M]\) a strongly pure ideal of \(R\) (Th. 4-11). Another consequence of theorem(4-11) is the following.

**Corollary**: Let \(R/I\) be an integral domain. \(M\) is multiplication \(R\)-module with strongly pure annihilator.

(1) If \(I\) is strongly pure ideal of \(R\), and \(N\) is a strongly pure submodule of \(M\). Then \(IN\) is a strongly pure submodule of \(M\).

(2) If \(N\) and \(L\) are strongly pure submodules of \(M\), then \(N+L\) is also strongly pure submodule of \(M\).

(3) If \(N\) and \(L\) are strongly pure submodules of \(M\), then \(N\otimes L\) is strongly pure submodule of \(M\otimes M\).

**Proof**: (1) As \(I\) is strongly pure ideal of \(R\), and \(N\) is strongly pure submodule of \(M\), then by theorem(4-11), we infer that each ideal \(I\) and submodule \(N\) are multiplication and idempotent ideal of \(R\) (sub module of \(M\)). Hence \(IN\) is multiplication [6, Corollary of theorem 2]. Moreover \(IN\) is idempotent. By theorem 3-11, we infer \(IN\) is strongly pure submodule of \(M\).

(2) Let \(x\in N+L\), there exists \(a\in N\) and \(b\in L\) such that \(x=a+b\).

By theorem(4-11) \(Ra=[N:M]a\) and \(Rb=[L:M]b\), and hence \(Rx=R(a+b)\subseteq Ra+Rb=[N:M]a+[L:M]b\subseteq((N+L):M](a+b)=[(N+L):M]x\).

And by theorem(4-11), \(N+L\) is strongly pure submodule of \(M\).

(3) Let \(a\in N\) and \(b\in L\). Then \(Ra=[N:M]a\) and \(Rb=[L:M]b\) and hence \(R(a\otimes b)=Ra\otimes Rb=[N:M]a\otimes [L:M]b=[N:M][L:M](a\otimes b)\). It is easy to check that \([N:M][L:M]\subseteq[a\otimes b:M\otimes M]\). Implies that \(R(a\otimes b)\subseteq[N\otimes L:M\otimes M](a\otimes b)\subseteq R(a\otimes b)\), so
that \( R(a \otimes b) \subseteq [N \otimes L : M \otimes M](a \otimes b) \). By theorem (4-11), we infer that \( N \otimes L \) is strongly pure sub module of \( M \otimes M \). Finally, we get the following results.

**Proposition** : Let \( M \) be an \( R \)-module, \( N \) and \( L \) are sub modules of \( M \) such that \( L \) is sub module of \( N \). If \( N \) is strongly pure sub module of \( M \), then \( N/L \) is strongly pure sub module of \( M/L \).

**Proof** : Since \( N \) is strongly pure sub module of \( M \), then \( NI = N \cap MI \), for some finitely generated prime ideal \( I \) of \( R \). So \( I(M/L) \cap N/L = IM/L \cap N/L = IM \cap N/L = IN/L = I(N/L) \). Thus \( N/L \) is a strongly pure sub module of \( M/L \).

### 5. STRONGLY REGULAR R-MODULES

In this section we introduce a generalization for regular \( R \)-module concept namely strongly regular \( R \)-module. Recall that an element \( x \in M \) is called regular if there exists an \( R \)-module homomorphism \( \theta : M \to R \), such that \( \theta(x)x = x \), [4]. If every element of \( M \) is regular, we say that \( M \) is regular \( R \)-module, [4].

And an \( R \)-module \( M \) is regular if for each \( x \in M \), and each \( r \in R \), there exists \( t \in R \), such that \( x r = x r t r \), [12]. First, we start this section by the following definition.

**Definition** : Let \( M \) be an \( R \)-module. An element \( x \in M \) is called strongly regular if there exists an \( R \)-module homomorphism \( \theta : M \to R \), such that \( \theta(x)x = x \) where \( \theta(x) \) is strongly regular element in a ring \( R \).

**Definition** : An \( R \)-module \( M \) is called strongly regular if every element of \( M \) is strongly regular.** Also** : \( M \) is strongly regular \( R \)-module if for every element \( x \in M \), and each \( r \in R \), there exists a prime element \( p \in R \), such that \( xr = x r p r \).

**Remark** : Every sub module of strongly regular \( R \)-module is strongly pure.

**Proof** : Let \( N \) be a sub module of \( M \), and let \( I \) be an ideal of \( R \). It is clear that \( IN \subseteq N \cap IM \). Conversely, let \( x \in N \cap IM \), then \( x = \sum_{i=1}^{k} r_i x_i \), where \( r_i \in I \) and \( x_i \in M \). Since \( M \) is a strongly regular \( R \)-module, hence \( x \) is strongly regular element. Thus there exists an \( R \)-module
homomorphism $\theta : M \rightarrow R$, such that $x = \theta(x)x$, so $\theta(x) = \sum_{i=1}^{n} r_{i}\theta(x_{i})$ and $x = \theta(x)x = \sum_{i=1}^{n} r_{i}\theta(x_{i})x$.

And since $x \in N$, hence $x = \sum_{i=1}^{n} r_{i}\theta(x_{i})x \in IN$. Thus $N \cap IM \subseteq IN$. Therefore $N$ is strongly pure sub-module of $M$.

Next, we want to study the relation between strongly regular ring and strongly regular module, by the following proposition.

**Proposition**: $R$ is strongly regular ring if and only if $R$ is strongly regular $R$-module.

**Proof**: Let $R$ be a strongly regular ring, and let $x \in R$. Thus there exists prime element $p \in R$, such that $x = xp_{x}$. Now defined a function $\theta : R \rightarrow R$, by $\theta(x) = xp$, for each $x \in R$, then $\theta(x)x = xp_{x}$, so $\theta(x)x = x$. Thus $R$ is strongly regular $R$-module.

Conversely; Let $R$ be strongly regular $R$-module, and let $x \in R$, there exists an $R$-module homomorphism $\theta : R \rightarrow R$, such that $x = \theta(x)x$, where $\theta(x)$ is strongly regular element of $R$. Since $\theta(x) = \theta(1)x = \theta(1)x$, so $xp_{x} = x\theta(1)x$. Therefore $R$ is strongly regular ring.

**Proposition**: If $M$ is strongly regular $R$-module, and divisible over an integral domain $R$, then every sub-module of $M$ is divisible.

**Proof**: Let $N$ be a sub-module of $M$, and let $0 \neq r \in R$, we show that $rN = N$. By remark (5-3) $N$ is strongly pure, so $rN = N \cap rM$. We show that $rN = N \cap rM$, if $x \in N \cap rM$, then $x = rm$, since $x$ is a strongly regular element of $M$, there exists an $R$-module homomorphism $\theta : M \rightarrow R$ such that $x = \theta(x)x$, and so $x = \theta(x)x = r\theta(m)x$, as $x \in N$ this implies that $x \in N$, so $N \cap rM \subseteq rN$, hence $N \cap rM = rN$. As $rM = M$, we see that $rN = N$, so $N$ is divisible.

**Proposition**: For every strongly regular $R$-module $M$, we have $J(R)M = 0$.

**Proof**: Since $M$ is strongly regular $R$-module, thus every sub-module of $M$ is strongly pure (Remark 5-3). Let $J(R)M \neq 0$, therefore there exists $x \in J(R)M$, so $Rx$ is strongly pure sub-module of $M$, thus $Rx \cap J(R)M = JRx$. Then $Rx = J(R)Rx$, and by Nakayama’s Lemma $Rx = 0$, so $x = 0$, and hence $J(R)M = 0$. 

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REFERENCES


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